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SOLUTION OF CONTACT PROBLEMS WITH KNOWN GREEN'S FUNCTION*

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An iteration algorithm is proposed for solving contact problems with previously unknown contact zone by taking account of friction in the contact zone, when the adhesion and slip zone interface is also unknown. Stability of the algorithm is investigated, and examples are presented.

Let us consider the problem of contact between a deformable body and an absolutely rigid smooth stamp. We assume that a function $k(x, y)$ is constructed which permits determination of the displacement $u(x)$ on the body surface by means of the contact pressure $p(y)$ given on this same surface. The classical scheme for solving the contact problem is to pass to the integral equation

$$\int_{S_c} k(x, y)\, p(y)\, dS_y = \varphi(x) \tag{0.1}$$

where the right side $\varphi(x)$ is determined by the initial gap between the body and the stamp. In general, the domain S_c is unknown, depending on the external effects, and is determined (for smooth stamps) from the condition that the contact pressure vanishes on the boundary of S_c.

The kernel $k(x, y)$ of equation (0.1) is singular for plane and spatial problems of elasticity theory, and often has a very complex form /1,2/ which makes the construction of analytic solutions and the utilization of numerical methods difficult. However, the main difficulty is that the problem of determining the function $p(y)$ from equation (0.1) is incorrectly formulated /3/. The method proposed below is free of this disadvantage and, moreover, permits easy determination of the true contact zone.

1. Description of the method. The method is based on one of the results in /4/, which is that the solution of the contact problem posed above is equivalent to seeking the saddle point of the following functional:

$$L(v, p) = J(v) + \int_{S_c} p\,(v_N - \delta_N)\, dS \tag{1.1}$$

where $J(v)$ is the energy functional, S_c is the greatest possible contact zone, δ_N is the initial gap, v is the displacement field, v_N is the normal displacement in the contact zone (see more accurate definitions in /5/). The saddle point (minimum in v and maximum in p) of the functional (1.1) is sought under the additional constraint

$$p(x) \leqslant 0 \tag{1.2}$$

The following modification of the Arrow–Hurwitz (Udzawa) method is hence used /6/:

1) The contact pressure distribution is given in the zero-th approximation $p = p^0(x)$;

2) The problem of minimization of the functional (1.1) in v is solved, whereupon the zero-th approximation of the displacement field $u = u^0(x)$ is determined;

3) For a fixed $u^0(x)$ the functional (1.1) is maximized in p by a gradient method with the projection in the set (1.2), which results in the following formula for the contact pressure in a first approximation

$$p^1 = P_\sigma\{p^0 - \rho_0(u_N^0 - \delta_N)\}, \quad P_\sigma\{p\} = \begin{cases} 0, & p > 0 \\ p, & p \leqslant 0 \end{cases} \tag{1.3}$$

where P_σ is the orthogonal projection operator in the set (1.2), and ρ_0 is a numerical parameter governing the step length.

If it is assumed that the function $k(x, y)$ is known, then stage 2) of this algorithm is realized exactly by means of the formula

$$u_N^0(x) = \int_{S_c} k(x, y)\, p^0(y)\, dS_y \equiv k\{p^0\} \tag{1.4}$$

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Calculations by means of (1.4) are fraught with certain difficulties of a technical nature, but not in principle.

2. Dual formulation of the method of solution. By the method of /4/ (Sect.2), it can be shown that the problem under consideration here about the contiguity of a deformable body to an absolutely rigid stamp is equivalent to the following problem for seeking the saddle point

$$\max_{\sigma^*\in M^*}\min_{v\in K}\left\{-\frac{1}{2}\int_\Omega A_{ijkh}\sigma_{kh}\sigma_{ij}\,d\Omega+\int_{S_u}\sigma_{ij}\nu_j g_i\,dS+\int_{S_c}\sigma_N v_N\,dS\right\} \qquad (2.1)$$

$$M^*=\{\sigma^*\mid \operatorname{div}\sigma^*+\rho F=0;\ \sigma_{ij}\nu_j\,|S_c=P_i;\quad \sigma_T\,|_{S_c}=0\} \qquad (2.2)$$
$$K=\{v\mid v_N(x)\leqslant\delta_N(x),\ x\in S_c\}$$

Here σ_{ij} are stress tensor components, A_{ijkh} are components of the compliance tensor, S_u is the part of the body surface where the displacements g_i are given while the forces P_i are given on S_c, and the condition $\sigma_T=0$ means no friction.

Realizing the idea of seeking the saddle point used in Sect.1, we arrive at the following algorithm (which it is natural to call the dual to the algorithm in Sect.1):

1) The normal displacement distribution on S_c is given in a zero-th approximation: $u_N=u_N^0(x)$;

2) The problem of maximizing the functional (2.1) in σ^* is solved, its exact solution reduced to solving an elasticity theory problem with the following boundary condition on S_c:

$$u_N=u_N^0(x),\quad \sigma_T(x)=0,\quad x\in S_c \qquad (2.3)$$

3) A step is taken in the direction of the most rapid decrease in the functional (2.1) with respect to v, a correction to u_N on S_c

$$u_N^{r+1}=P_u\{u_N^r-\rho_r\sigma_N^r\} \qquad (2.4)$$

where P_u is the orthogonal projection operator in the set K, ρ_r is a dimensional parameter controling the step length, and r is the number of the iteration.

To realize stage 2) of this algorithm, it is necessary to have an operator connecting the force on the body surface with the displacements of points on this same surface, the inverse to (1.4).

3. On the stability of the algorithms. Let the left side of the inequality, the non-penetration condition, be denoted by $\Phi(u)$, and let us rewrite the formula (1.3) as follows (s is the number of the approximation):

$$\frac{p^{s+1}-p^s}{\rho_s}=\frac{1}{\rho_s}\left[P_\sigma\{p^s+\rho_s\Phi(u^s)\}-p^s\right]$$

Performing the passage to the limit here as $\rho_s\to 0$, we obtain a "differential"equation for p

$$p^{\cdot s}=P_\sigma\{\Phi(u^s)\}$$

By using (1.4), we find (we omit the superscript s)

$$p^\cdot=P_\sigma\{\Phi(k\{p\})\} \qquad (3.1)$$

The algorithm (1.3) can evidently be treated as the simplest difference scheme for the solution of the "ordinary" differential equation (3.1). By approximating the derivative $p^\cdot$ by more exact difference relations, we arrive at other iteration schemes for solving the contact problem.

Let us emphasize that (1.3) does not describe the physical process but the motion to the saddle point along a certain growth trajectory of the functional $L(v,p)$ with respect to p. By constructing the trajectory of such motion from more exact considerations, differential equations can be obtained with derivatives of order higher than the first, and iteration procedures of higher accuracy can naturally be obtained which are perhaps more advantageous, but this question requires a separate investigation.

By using the known results about the qualitative properties of the operator k and theorems on the stability of solutions of differential equations in Banach spaces /7/, the question of the stability of the iteration process with respect to roundoff errors and other perturbations associated with the error of discretization can be investigated. In particular, by using known theorems on the spectrum of the operator k, we have an assertion concerning the the stability of the procedures proposed above for contact problems with a constant contact zone, including mixed problems (let us again note that we speak about the stability of the solutions of equation (3.1)). For problems with a variable contact zone, the operator P_σ $\{\Phi(k\{p\})\}$ is nonlinear; however it is non-negative determined, which affords a possibility for affirmative solution, in principle, of the question of stability also in this case.

4. **Examples.** 1°. Let us examine the classical problem of the insertion of an absolutely rigid ball in an elastic isotropic half-space and in the example of this problem we study the question of the rate of convergence of an Udzawa-type iteration procedure, the behavior of the sequence of approximate solutions, the difference in the solutions due to the difference in the possible non-penetration condition.

We perform discretization by the influence matrix method (A.A. Il'iushin). To do this, we enclose the greatest possible contact zone in a square, we select the axes Ox_1, Ox_2 along the sides of the square, and direct the Ox_3 axis deep into the half-space. We partition the square by lines parallel to the axes Ox_1 and Ox_2

$$x_1^i = (i-1)h, \quad x_2^j = (j-1)h, \quad 1 \leqslant i, j \leqslant M+1$$

and in each little square (ij) (we consider the number of the square to agree with the number of the corner, the apex closest to the origin) we approximate the contact interaction force by the constant p_β^{ij}, $\beta = 1,2,3$; we combine the set of all such constants into the vector $\{p\}$, and we combine the set of displacements $u_\beta^{kl} = u_\beta(x_1^k, x_2^l, 0)$, $\beta = 1,2,3$ into the vector $\{u\}$. By using the Boussinesq and Cerruti solutions for a half-space, we construct the matrix $[B]$ connecting the vectors $\{p\}$ and $\{u\}$

$$\{u\} = [B]\{p\}$$

The elements of the matrix $[B]$ are calculated explicitly; the details of the calculations and the methods of shaping and storing the matrix $[B]$ are published in /8/.

The rate of convergence of the sequence of approximate solutions obtained by using the algorithm 1)—3) of Sect.1 is illustrated in Figs.1 and 2. The function $p = 1$ is selected as the zero-th approximation. Here N is the number of the iteration in Fig.1, and $\|p\| = \sum_{k,l}|p_3^{kl}|$. It is seen that $\|p\|$ (the dashed line) and the pressure $p \equiv p_3$ at the center point (0.5,0.5,0) (solid line) are rapidly stabilized, where p tends non-monotonically to the exact solution. Contact pressure distributions over the diameter are shown in Fig.2, where the number of the curve is the number of the iterations. It is seen that the pressure varies sufficiently sharply in a strip adjacent to the edge of the contact zone, despite the stabilization of $\|p\|$. Therefore, the local condition for stopping the iteration process (ε is a given positive number)

$$\max_{(ij)} |p_{N+1}^{ij} - p_N^{ij}| < \varepsilon$$

is more preferable than global conditions of the type $\|p_{N+1} - p_N\| < \varepsilon$ or $|\|p_{N+1}\| - \|p_N\|| < \varepsilon$.

As has been established in /5/, the non-penetration conditions

$$\Psi(x) + u(x)\cdot\nabla\Psi(x) \geqslant 0, \quad u_N(x) \leqslant \delta_N(x) \tag{4.1}$$

(δ_N is a segment normal to the undeformed boundary of the body and the stamp, and u_N is the projection of u on the normal to the body) are equivalent in the sense of asymptotic accuracy, as $|u|$ and the gap δ_N tend to zero.

The contact pressure diagrams obtained are displayed in Fig.3; the solid curve corresponds to the first of the conditions (4.1), and the dashes to the second. These curves differ by 15% at the center of the contact zone; such a noticeable effect is explained by the comparatively large depth of insertion $0.1627R$ (R is the radius of the ball).

The displacement distributions corresponding to conditions (4.1) are displayed in Fig.4 (the solid curve corresponds to the first of conditions (4.1) and the dashes to the second, the normal displacements are large in absolute value). The difference between the normal displacement diagrams is explained by the fact that the displacements squared are discarded in the derivation of the first of conditions (4.1) (in both cases the non-penetration conditions are satisfied to three significant figure accuracy).

2°. Let us consider the problem of a ball rolling with friction on the boundary of a half-space with constant linear and angular velocities $v_c^\cdot$ and ω_c in direction and magnitude, where the vector ω_c is assumed perpendicular to the vector $v_c^\cdot$. Let us also give the depth of submersion δ of the ball; by knowing $v_c^\cdot, \omega_c, \delta$, we are required to determine the contact zone and the contact interaction force. To solve the problem we apply the principle of possible velocities, in conformity with which we have the equation

$$\int_\Omega a_{ijkh}\varepsilon_{kh}(u)\varepsilon_{ij}(\delta u^\cdot)\,d\Omega - \int_{S_c}\sigma_N\delta u_N^\cdot\,dS - \int_{S_c}\sigma_T\cdot(v_T^\cdot - u_T^\cdot - u_{T_c}^\cdot)\,dS = 0 \tag{4.2}$$

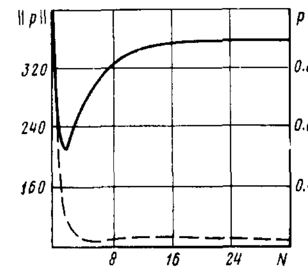

Fig.1

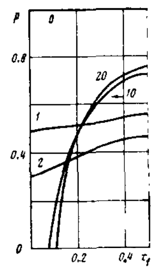

Fig.2

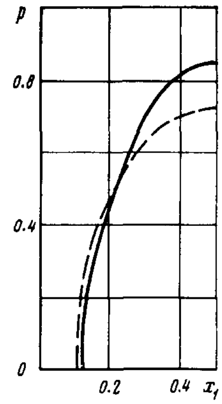

Fig.3

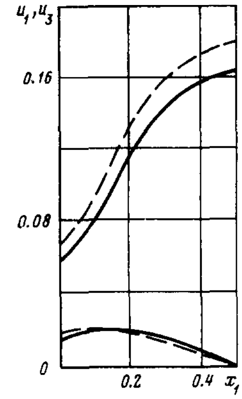

Fig.4

where $u_{Tc}^{\cdot}$ is the projection of the velocity of a point of the ball surface on the tangent plane. Equation (4.2) is written in a moving coordinate system whose origin is at the center of the ball, the Ox_1 axis agrees with the direction of the vector $-v_c^{\cdot}$, the Ox_2 axis with ω_c, and the Ox_3 axis is directed deeply into the half-space. The non-penetration condition that agrees with the first of the conditions (4.1) was used in the calculations so that σ_N in (4.2) is the projection of the surface force vector on the normal to the ball, σ_T onto the tangential plane, $v_T^{\cdot}$ is the tangential component of the possible velocity, and $u_T^{\cdot}$ is the true velocity.

It has been established /9/ that (*)

$$u_T^{\cdot} = |v_c^{\cdot}|\,(\partial u/\partial x_1) \tag{4.3}$$

In the calculations, the derivative $\partial u/\partial x_1$ was replaced by the ratio of the finite differences.

To solve (4.2) with an associated Coulomb friction law, non-penetration condition, and condition of no tensile forces in the contact zone, an algorithm was used which is a modification of the algorithm in /10/, whose convergence is established also as in /10/:

1) Given are contact interaction forces in the zero-th approximation

$$\sigma_N = \sigma_N^0(x),\quad \sigma_T = \sigma_T^0(x);$$

2) The elasticity theory problem is solved with the boundary conditions of the form

$$\sigma_{ij}\nu_j = \sigma_N^0\nu_i + (\sigma_T^0)_i;$$

on S_c; actually, calculations are performed by the formula $\{u\} = [B]\{p\}$, whereupon the displacement field $u = u^0(x)$ is determined;

3) The contact interaction forces are corrected by a formula of the type (1.3) for σ_N and

$$\sigma_T^1 = P_T\{\sigma_T^0 - \rho_{T0}(u_T^{\cdot 0} - u_{T_c}^{\cdot})\} \tag{4.4}$$

$$P_T\{\sigma_T\} = \begin{cases} \sigma_T, & |\sigma_T| \leqslant f|\sigma_N| \\ (\sigma_T/|\sigma_T|)f|\sigma_N|, & |\sigma_T| > f|\sigma_N| \end{cases} \tag{4.5}$$

where $u_T^{\cdot}$ in (4.4) is evaluated by means of (4.3).

The initial data are the length of a side of the square $l = 1$, the radius of the ball $R = 1$, $M = 16$, the shear modulus $\mu = 1$, the Poisson's ratio $\nu = 0.3$, $\delta = 0.1627R$, $|\omega_c| = |v_c^{\cdot}|/R$ (the absolute value of $v_c^{\cdot}$ evidently does not influence the result), the friction coefficient $f = 0.1$.

Some of the results obtained are given in Fig.5. The solid line is the displacement distribution over the diameter of the contact spot in the motion direction, the dashed line is the distribution of the relative velocities $v_T^{\cdot} = u_T^{\cdot} - u_{Tc}^{\cdot}$ on the line $x_1 = 0.5$; there is evidently a cohesion zone at the center of the contact spot and two slip zones along the edges. Let us note that a lead zone, particle drift ahead of the motion, was obtained in certain modifications of the computations, however this effect is quite weak and the power of the electronic computer (EC-1022) turned out to be insufficient for a confident prediction.

*) Gol'dshtein, R.V., A.F. Zazovskii, A.A. Spektor, and R.P. Fedorenko. Solution of three-dimensional contact problems of rolling with slip and cohesion. Preprint No.134, Inst. Problem of Mechanics, Acad. Sci. USSR, Moscow, 1979.

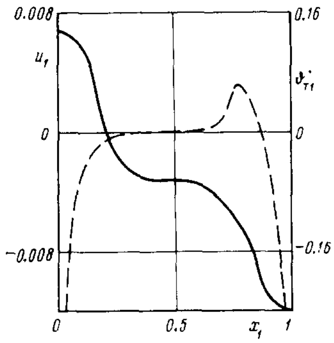

Fig.5

Let us also note that the convergence of the method used in the rolling problem is slow, and stabilization of the results was achieved successfully only for $N \sim 150-200$. The convergence can visibly be accelerated by simultaneous utilization of the matrix $[B]$ and the matrix relating the derivatives with respect to x_1 of the solution on the boundary to the vector $\{p\}$; such a matrix was constructed earlier (*).

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*) See previous footnote, p.217.